Series and Sequences

1 Introduction

Sequences and series are all about finding and exploting patterns. Each term in a sequence is based in some way to terms prior to it. In this Math League session, we focus on different types of sequences and their patterns and learn different tips and techniques for working with problems of this type.

2 Arithmetic Series

Arithmetic series are ones that you should probably be familiar with. As a reminder, in an arithmetic sequence or series the each term differs from the previous one by a constant. An example of an arithmetic sequence is $1, 3, 5, 7, 9, \ldots$.

Theorem 1 (Gauss)**.** *Let S denote the sum of the terms of an n-term arithmetic sequence with first term a and common difference d. Then*

$$S = \frac{1}{2}n\left[2a + (n-1)d\right].$$

Proof. Flip the sum over on itself and add it to the original series, as such:

$$\begin{array}{rcccccccc} S & = & a & + & (a+d) & + & (a+2d) & +\cdots+ & [a+(n-1)d] \\ S & = & [a+(n-1)d] & + & [a+(n-2)d] & + & [a+(n-3)d] & +\cdots+ & a \\ \hline 2S & = & [2a+(n-1)d] & + & [2a+(n-1)d] & + & [2a+(n-1)d] & +\cdots+ & [2a+(n-1)d]. \end{array}$$

Since there are n copies of $2a + (n-1)d$ in the bottom sum, we have

$$2S = n[2a + (n-1)d] \implies S = \tfrac{1}{2}n[2a + (n-1)d]$$

as desired. ■

Note. *If the last term z is given, this expression may be simplified to $S = \frac{n(a+z)}{2}$.*

Example 1 (AMC 10B/12B 2004)**.** *A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?*

Solution. Note that the number of cans on each of the n rows forms an arithmetic sequence with first term 1 and common difference 2. The total number of cans on n rows is thus

$$\frac{1}{2}n[2 + (n-1)\cdot 2] = \frac{1}{2}n[2n] = n^2.$$

Setting this equal to 100 gives $n = \boxed{10}$. ■

Example 2 (AMC 10B 2002)**.** *Suppose that $\{a_n\}$ is an arithmetic sequence with*

$$a_1 + a_2 + \cdots + a_{100} = 100 \qquad \text{and} \qquad a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

Solution. Let $d = a_2 - a_1$ be the common difference of the arithmetic sequence. Note that $a_{101} = a_1 + 100d$, $a_{102} = a_2 + 100d$, and so on. Substituting these expressions into the second equality gives

$$\begin{aligned} (a_1 + 100d) + (a_2 + 100d) + \cdots + (a_{100} + 100d) &= 200 \\ (a_1 + a_2 + \cdots + a_{100}) + 10000d &= 200 \\ 100 + 10000d &= 200 \\ d &= \boxed{0.01}. \end{aligned}$$

■

Now try your hands at the following problem. It is not hard; however, it sheds light on a very powerful tool when dealing with arithmetic sequences (and many other branches of math as well).

Problem 1 (AMC 12A 2007)**.** *Let a, b, c, d, and e be five consecutive terms in an arithmetic sequence, and suppose that $a + b + c + d + e = 30$. Which of the following can be found?*

(A) a **(B)** b **(C)** c **(D)** d **(E)** e

3 Geometric Series

Geometric series are a sort of counterpart to arithmetic series. Instead of the difference between two adjacent terms being constant, the *quotient* between two adjacent terms is constant. An example of a geometric sequence is $1, 2, 4, 8, 16, 32, 64, \cdots$. In that sequence, each term is double the previous one.

There also exists a formula for the sum of a finite geometric series, and it is derived in a somewhat-similar way.

Theorem 2. *Let S be the sum of a n-term geometric series with first term a and common ratio r. Then*

$$S = \frac{a(1-r^n)}{1-r}.$$

Proof. Note that $S = a + ar + ar^2 + \cdots + ar^{n-1}$. Remark that multiplying by r gives $rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$. When this second equality is subtracted from the first, almost all of the terms cancel out. What is left is

$$S - Sr = a - ar^n = a(1-r^n) \implies S = \frac{a(1-r^n)}{1-r},$$

as desired. ■

Corollary 1. *If the geometric series is an infinite series and $|r| \le 1$ (i.e. the series converges), then the formula is simplified to $S = \frac{a}{1-r}$.* ***Know this!***

Example 3. *Let p and q be real numbers with $|p| < 1$ and $|q| < 1$ such that*

$$p + pq + pq^2 + pq^3 + \cdots = 2 \qquad \text{and} \qquad q + qp + qp^2 + qp^3 + \cdots = 3.$$

Find $100pq$.

Solution. Note that by the formulae for geometric series the first equation can be rewritten as $\frac{p}{1-q} = 2$ and the second can be rewritten as $\frac{q}{1-p} = 3$. This gives the system of equations

$$\begin{cases} p &= 2 - 2q, \\ q &= 3 - 3p. \end{cases}$$

Solving this gives $(p, q) = (\frac{4}{5}, \frac{3}{5})$, and so $100pq = 100(\frac{12}{25}) = \boxed{48}$. ■

Now try the following problem:

Problem 2 (Winter OMO 2011)**.** *If*

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{8x^4} + \frac{1}{16x^5} + \cdots = \frac{1}{64},$$

and x can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, find $m + n$.

4 Telescoping Series

Telescoping Series are a new type of beast that most of you have probably never seen before. Imagine a normal household telescope when fully expanded. Now say that a novice astronomer is going to put his telescope away. What does he do with it? He collapses it, greatly shrinking its length. This is what we do with telescoping sums.

Example 4 (Classic)**.** *Evaluate the sum*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}.$$

Solution. Evaluating the first several partial sums of this series, we get the sequence $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and so on, which suggests the answer is $\frac{99}{100}$.[1] Indeed, this cancellation is not coincidental: if one writes $\frac{1}{2} = 1 - \frac{1}{2}$, $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$, $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, and so on, then the sum becomes

$$\left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{99}} - \frac{1}{100}\right) = 1 - \frac{1}{100} = \boxed{\frac{99}{100}}.$$

■

Here is an example of a more difficult telescoping summation.

Example 5 (Purple Comet HS 2004)**.** *Define $a_k = (k^2+1)k!$ and $b_k = a_1 + a_2 + a_3 + \cdots + a_k$. Let*

$$\frac{a_{100}}{b_{100}} = \frac{m}{n}$$

where m and n are relatively prime natural numbers. Find $n - m$.

Solution. Note that

$$\begin{aligned}(k^2+1)k! &= k(k \cdot k!) + k! = k\left[(k+1)! - k!\right] + k! \\ &= k(k+1)! - k \cdot k! + k! = k(k+1)! - k!(k-1).\end{aligned}$$

Hence for all positive integer k we have

$$\begin{aligned}b_k &= a_1 + a_2 + a_3 + \cdots + a_k \\ &= \cancel{1 \cdot 2!} - 1! \cdot 0 + \cancel{2 \cdot 3!} - \cancel{2! \cdot 1} + \cancel{3 \cdot 4!} - \cancel{3! \cdot 2} + \cdots + k(k+1)! - \cancel{k!(k-1)} \\ &= k(k+1)!\end{aligned}$$

Thus

$$\frac{a_k}{b_k} = \frac{(k^2+1)k!}{k(k+1)!} = \frac{k^2+1}{k(k+1)} = \frac{k^2+1}{k^2+k}.$$

Substituting $k = 100$ gives $\frac{a_{100}}{b_{100}} = \frac{10001}{10100}$, and the requested answer is $10100 - 10001 = \boxed{99}$. ■

Telescoping expressions are not just limited to sums. Nasty products as well can be significantly simplified through rampant cancellation. Try the following problem on your own:

Problem 3 (HMMT 2006)**.** *Find*

$$\frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{4^2}{4^2-1} \cdot \cdots \cdot \frac{2006^2}{2006^2-1}.$$

5 Problem Set A

1. [AMC 12A 2014] The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

2. [Math League HS 2011-2012] What is the average of 2012 consecutive positive integers whose sum is 2012^3?[2]

3. [Purple Comet HS 2012] In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds 1000 kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

[1] This type of reasoning if used in a formal solution is sometimes colloquially referred to as *Engineer's Induction.* While it is okay for short-answer tests where time is a crucial factor, it is frowned upon otherwise. So always try and search for mathematical rigor in any patterns you find.

[2] Please excuse the fact that such a sequence of integers does *not* actually exist.

4. [Math League HS 2012-2013] Five positive integers are written in increasing order, and the difference between adjacent terms is constant. If the sum of the integers is 540, what is the maximum possible value of the largest integer?

5. [AHSME 1993] Let $a_1, a_2, ..., a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17$$

and

$$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 77.$$

If $a_k = 13$, what is k?

6. [HMMT 2003] Find the value of $\dfrac{1}{3^2+1} + \dfrac{1}{4^2+2} + \dfrac{1}{5^2+3} + \cdots$.

7. [CEMC] Prove that if a, b, c, and d are four consecutive terms in a geometric sequence, then

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2.$$

8. [HMMT 2002] Determine the value of the sum

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{29}{14^2 \cdot 15^2}.$$

6 Problem Set B

9. [AIME 1984] Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if $a_1,\ a_2,\ a_3,\ \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$.

10. [Math League HS 2000-2001] Evaluate

$$\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) + \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\right) + \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right) + \cdots.$$

More formally, what is the sum of all fractions of the form $\frac{1}{(m+1)^{n+1}}$, where m and n range over the positive integers?

11. [AIME 1989] If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k.

12. [AMC 12A 2014] Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of c?

13. [Putnam 2013] For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

★ 14. [AoPS] If a, b, and c form an arithmetic progression, and

$$a = x^2 + xy + y^2,$$
$$b = x^2 + xz + z^2,$$
$$c = y^2 + yz + z^2,$$

where $x + y + z \neq 0$, prove that x, y, and z also form an arithmetic progression.

7 Problem Set C

"If you're doing something easy enough that you're not seriously scared you're going to screw up, it's probably not the most meaningful use of your time."

~Evan Chen (v_Enhance)

15. [USAMTS 1999] Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}.$$

★ 16. [AIME 2003] In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

★ 17. [AIME 2004] A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms $a_{2n-1}, a_{2n}, a_{2n+1}$ are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

★ 18. [Mandelbrot] Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Find the value of the infinite sum

$$\frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \cdots + \frac{F_n}{3^n} + \cdots.$$

(Hint: consult the proof of Theorem 2.)

★★ 19. [Fall OMO 2013] The real numbers $a_0, a_1, \ldots, a_{2013}$ and $b_0, b_1, \ldots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \ldots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute

$$\sum_{k=1}^{2013} (a_k b_{k-1} - a_{k-1} b_k).$$